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MEMORANDUM

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ANALYSIS OF A NEW FORMALISM IN
PERTURBATION THEORY

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PREFACE

The Project RAND research program consists in part of basic supporting studies in mathematics. This Memorandum analyses a new method for the solution of equations of the form

$$u = f + \lambda T(u) ,$$

where T is a linear transformation. Such equations are of great importance in mathematical physics.

SUMMARY

A large number of problems of mathematical physics may be reduced to the solution of the equation

$$u = f + \lambda T(u) ,$$

where T is a linear transformation. The present paper reduces a recent proposal to treat such problems by means of continued fractions to a more tractable method involving series, and demonstrates the convergence of the resulting series over a larger domain than the classical Neumann series.

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ANALYSIS OF A NEW FORMALISM IN PERTURBATION THEORY

1. INTRODUCTION

A new method has recently been suggested by Bellman and Richardson [1] for studying equations of the form

$$(1.1) \quad u = f + \lambda T(u) ,$$

where f is a known function, u is unknown, λ is a parameter, and T is a linear transformation. This suggestion is based on the observation that, if u is a solution to (1.1), then

$$(1.2) \quad \frac{T^n(u)}{T^{n+1}(u)} = \frac{T^n(f)}{T^{n+1}(f)} + \lambda \left[1 - \frac{T^n(f)/T^{n+1}(f)}{T^{n+1}(u)/T^{n+2}(u)} \right] .$$

Thus, if the value of $T^n(u)/T^{n+1}(u)$ can be determined, the value of $u/T(u)$ can be determined recursively, and by combining this with (1.1), the value of u itself may be computed. If we make the assumption that $T^n(u)/T^{n+1}(u)$ equals $T^n(f)/T^{n+1}(f)$, then carrying out the above computation gives a sequence of convergents to a certain continued fraction. One could analyze the method by considering this continued fraction, but in what follows we shall show that it is quite easy to evaluate and analyze the convergents themselves.

2. PRECISE DEFINITION OF THE CONVERGENTS

Characterize the function u_n as follows:

$$(2.6) \quad \frac{T^k(u_k)}{T^{k+1}(u_k)} = \frac{T^k(f)}{T^{k+1}(f)},$$

$$\frac{T^{k-1}(u_k)}{T^k(u_k)} = \frac{T^{k-1}(f)}{T^k(f)} + \lambda \left[1 - \frac{T^{k-1}(f)/T^k(f)}{T^k(u_k)/T^{k+1}(u_k)} \right]$$

is equivalent to replacing $T^k(f)$ (in the formula of form (2.4) for u_{k-1}) with

$$(2.7) \quad \frac{[T^k(f)]^2 T^{k-1}(f)}{T^{k-1}(f) T^{k+1}(f) + \lambda [(T^k(f))^2 - T^{k-1}(f) T^{k+1}(f)]}.$$

If we carry out this substitution, we find that u_k also satisfies the formula (2.4), and thus the formula holds for all n .

3. PROPERTIES OF THE u_n

Theorem 3.1 If $T^{n+1}(f) = \mu T^n(f)$ for some $\mu \neq 1/\lambda$, then u_n is a solution to (1.1).

Proof: Substitute (2.4) directly into (1.1).

Theorem 3.2 Within the radius of convergence of the Neumann series, the sequence $\{u_n\}$ converges to a solution of (1.1).

Proof: The radius of convergence of the Neumann series is the absolute value of the eigenvalue of T having the smallest norm. Thus if λ is less than this eigenvalue we may, by taking n sufficiently large, make

the difference between u_n and the n^{th} partial sum of the Neumann series as small as we please. Thus, the sequence $\{u_n\}$ converges to the same limit as the Neumann series, which is a solution of (1.1).

Theorem 3.3 If $f = f_1 + f_2$, where T is continuous, $T(f_1) = \mu_1 f_1$ and $|T^k(f_2)| \leq \mu_2 |T^{k-1}(f_2)|$ (for $k=1, 2, \dots$). $\mu_1 > \mu_2 > 0$, $\lambda \neq 1/\mu_1$, $|\lambda| < 1/|\mu_2|$, then $\{u_n\}$ is a Cauchy sequence convergent to a solution of (1.1).

Proof: If k is sufficiently large, it is a matter of straightforward computation to show that

$$(3.1) \quad \left\| \frac{[T^k(f)]^2}{T^k(f) - \lambda T^{k+1}(f)} - \frac{\mu_1^{2k} f_1^2}{\mu_1^k f_1 - \lambda \mu_1^{k+1} f_1} \right\| \leq \frac{4\lambda \|f_2\| (\mu_1 - \mu_2)}{(1 - \lambda \mu_1)^2 \|f_1\|} \mu_2^k = c \mu_2^k.$$

Say that (3.1) holds for all $k > N$. Pick $m > n > N$.

Now consider

$$(3.2) \quad \frac{u_m - u_n}{\lambda^n} = \sum_{i=0}^{m-n-1} T^{n+i}(f) \lambda^i + \frac{\lambda^{m-n} [T^m(f)]^2}{T^m(f) - \lambda T^{m+1}(f)} - \frac{[T^n(f)]^2}{T^n(f) - \lambda T^{n+1}(f)}.$$

Simple computation shows that

$$\begin{aligned}
 (3.3) \quad s &= \sum_{i=0}^{m-n-1} T^{n+1}(f_1) \lambda^i + \frac{\lambda^{m-n} [T^m(f_1)]^2}{T^m(f_1) - \lambda T^{m+1}(f_1)} \\
 &\quad - \frac{[T^n(f_1)]^2}{T^n(f_1) - \lambda T^{n+1}(f_1)} = \left[\sum_{i=0}^{m-n-1} (\mu_1 \lambda)^i + \frac{(\mu, \lambda)^{m-n}}{1 - \mu_1 \lambda} \right. \\
 &\quad \left. - \frac{1}{1 - \mu_1 \lambda} \right] f_1 = 0 .
 \end{aligned}$$

Thus it follows, from (3.1) and the linearity of T , that

$$\begin{aligned}
 (3.4) \quad \frac{\|u_m - u_n\|}{\lambda^n} &= \frac{\|u_m - u_n - s\|}{\lambda^n} \\
 &\leq \sum_{i=0}^{m-n-1} |\mu_2 \lambda|^i |\mu_2|^n \|f_2\| + |\lambda|^{m-n} c |\mu_2|^m \\
 &\quad + c |\mu_2|^n \leq |\mu_2|^n \left[\frac{\|f_2\|}{1 - |\mu_2 \lambda|} + 2c \right] .
 \end{aligned}$$

Since $|\lambda \mu_2| < 1$ by hypothesis, this establishes that the sequence $\{u_n\}$ is a Cauchy sequence. Now to show that it converges to a solution of (1.1), we have only to show that for any $\epsilon > 0$, we can find an N such that if $n > N$, then

$$\|u_n - \lambda T(u_n) - f\| < \epsilon .$$

This is equivalent to showing that

$$(3.5) \quad |\lambda|^{n+1} \left\| \frac{T^{n+1}(f)T^n(f)}{T^n(f) - \lambda T^{n+1}(f)} - T \left[\frac{T^n(f)T^n(f)}{T^n(f) - \lambda T^{n+1}(f)} \right] \right\| < \epsilon$$

By applying the identity

$$(3.6) \quad \frac{T^n(f)}{T^n(f) - \lambda T^{n+1}(f)} = \frac{1}{1 - \lambda \mu_1} + \frac{\lambda T^{n+1}(f_2) - \lambda \mu_1 T^n(f_2)}{(1 - \lambda \mu_1)(T^n(f) - \lambda T^{n+1}(f))}$$

we reduce (3.5) to

$$(3.7) \quad \frac{|\lambda|^{n+1}}{|1 - \lambda \mu_1|} \left\| \frac{\lambda T^{n+1}(f_2) - \lambda \mu_1 T^n(f_2)}{T^n(f)/T^{n+1}(f) - \lambda} - T \left[\frac{\lambda T^{n+1}(f_2) - \lambda \mu_1 T^n(f_2)}{1 - \lambda T^{n+1}(f)/T^n(f)} \right] \right\|$$

But since by assumption $|\lambda|^n |T^n(f_2)|$ can be made as small as we please by taking n sufficiently great, it follows at once that (3.7) goes to zero as n becomes large. This concludes the proof.

4. EXAMPLES

A. If $T(f) = kf$, where k is some constant, then by Theorem 3.1, every u_n is an exact solution to (1.1) (provided $\lambda \neq 1/k$).

B. Let the transformation T be the matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, which has eigenvalues 1 and 3. Then the sequence $\{u_n\}$ will

converge by Theorem 3.3 to a solution of (1.1) if $|\lambda| < 1$, $\lambda \neq 1/3$. If f is an eigenvector of T , then each u_n will be an exact solution of (1.1) unless λ is the reciprocal of the corresponding eigenvalue. This example shows that $\{u_n\}$ need not converge if λ is greater than the reciprocal of the second largest eigenvalue.*

C. The formula may be applied to a wide variety of integral equations. Even when the Neumann series converges, the sequence (2.4) will often converge faster.

* The reader is doubtless aware that the eigenvalue of a matrix corresponds to the reciprocal of the eigenvalue of an integral operator (which is the sense in which we used the term in the proof of Theorem 3.2.)

REFERENCE

1. Bellman, Richard and John M. Richardson, "A New Formalism in Perturbation Theory Using Continued Fractions," Proc. Nat. Acad. Sci., USA, Vol. 48, 1962, pp. 1913-1915.